Stability of certain families of ideal magnetohydrodynamic equilibria

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The equations of ideal magnetohydrodynamic equilibria posses a number of symmetries that may be used to generate a family of hitherto unknown equilibria if there exists a foliation of the original one by magnetic surfaces. In addition to the possibility of producing analytic equilibria from old ones, this family is studied to find among its members those with minimal energy, those lasting longer under slightly resistive conditions, and those linearly stable. It is shown that in general none of these properties implies any other, thus clarifying the difference among these concepts.

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I. INTRODUCTION

Analytical magnetohydrodynamic (MHD) equilibria are notoriously difficult to find. The best known examples include those where the velocity lies in a plane and the magnetic field is orthogonal to it [1], plus a number of static configurations, including the axisymmetric ones satisfying the Grad-Shafranov equation, such as the Shafranov-Solov'ev family [2]; a recent class of nonaxisymmetric static equilibria is presented in Ref. [3]. In these circumstances, the method stated in Ref. [4] for generating new equilibria starting with an initial one possessing a foliation of magnetic surfaces is welcome. This family is an excellent tool to clarify a number of concepts on ideal MHD equilibria, such as the possible existence of attractive equilibria, the stability of minimal energy ones, and their relation with those lasting longer under slightly resistive conditions.

Let us begin by recalling the basic ideas in Ref. [4]. An ideal MHD equilibrium with velocity $\mathbf{V}$, magnetic field $\mathbf{B}$, density ρ, and total (kinetic plus magnetic) pressure P_* satisfies, after normalization of constants,

$$\rho\mathbf{V}\cdot\nabla\mathbf{V}-\mathbf{B}\cdot\nabla\mathbf{B}+\nabla P_*=\mathbf{0},$$

$$\nabla\times(\mathbf{V}\times\mathbf{B})=\mathbf{0},$$

$$\nabla\cdot\mathbf{B}=\nabla\cdot\rho\mathbf{V}=0. \tag{1}$$

Assume also that the plasma is incompressible, $\nabla\cdot\mathbf{V}=0$. This does not necessarily mean that the density is constant, although it is along streamlines: $\mathbf{V}\cdot\nabla\rho=0$. The fact that $\nabla\times(\mathbf{V}\times\mathbf{B})=\mathbf{0}$ means that $\mathbf{V}\times\mathbf{B}$ is locally the gradient of a scalar function ψ. ψ is unique, except by additive constants, in any simply connected subset of the domain Ω, and the level sets $\psi=$const form a foliation of this, provided $\mathbf{V}\times\mathbf{B}\neq\mathbf{0}$ at every point. Since the equilibria where $\mathbf{V}$ is collinear with $\mathbf{B}$ (or simply zero) are very important, they need a separate treatment. The main hypothesis is the existence in a subdomain U of Ω of a foliation of magnetic surfaces: i.e., there exists in U a function ψ such that $\nabla\psi\neq 0$ almost everywhere, and $\mathbf{B}\cdot\nabla\psi=\mathbf{V}\cdot\nabla\psi=0$. The normal vector $\nabla\psi$ is determined up to a factor at every point where $\mathbf{V}$ and $\mathbf{B}$ are not collinear. If this happens everywhere in U, the possibilities are manifold: $\mathbf{B}$ may be ergodic in U, a single field line being dense, so that ψ must be constant. In the opposite situation, there may exist a family of magnetic surfaces with two degrees of freedom, such as any cylindrical surface when the field is unidirectional; or there could be a foliation with the magnetic field being ergodic in almost every surface, such as the classical toroidal case of field lines winding up almost every torus (the irrational ones) ergodically. These cases are comparatively simple: magnetic surfaces may exist locally, but their extensions may have self-intersections in every conceivable way. Let us therefore simply admit the existence of a foliation, labeled ψ, in U such that the velocity and field are tangential to the surfaces. We could admit ψ failing to be smooth at some discrete points: e.g., when separate surfaces coalesce for a certain value of ψ. However, we need the level surfaces S_ψ to be continuous in the following sense: for any smooth function f in U, the function

$$\psi\to\int_{S_\psi} f\,d\sigma$$

is continuous. U may be immersed in Ω, or the boundary ∂U be part of $\partial\Omega$. It is known that islands of the structured field may exist in a medium of chaotic plasma, and this is a feature common to many Hamiltonian chaotic phenomena: see, e.g., Ref. [5]. Thus it is reasonable to admit the possibility that our domain U may not be the whole of Ω. Finally, we assume that the density ρ is constant on any level surface. This does not follow from the incompressibility of the plasma except when a single streamline is dense at the surface, but it holds in many important cases, e.g., in toroidal confinement.

II. THE FAMILY OF EQUILIBRIA

Let us recall from [4] the transformations that keep the state of equilibrium:

$$\rho_1=a(\psi)\rho,$$

$$\mathbf{B}_1=b(\psi)\mathbf{B}+c(\psi)\sqrt{\rho}\mathbf{V},$$

$$\mathbf{V}_1=\frac{c(\psi)}{a(\psi)\sqrt{\rho}}\mathbf{B}+\frac{b(\psi)}{a(\psi)}\mathbf{V},$$

$$P_{1*} = CP_*, \tag{2}$$

where a, b, c are functions of ψ and $b^2 - c^2 = C$ is constant in U; hereafter we will omit, for simplicity, the dependence on ψ of a, b, and c from our equations. So far we have essentially explained the contents of Refs. [3,4]. Taking the magnitude $\sqrt{\rho}\mathbf{V}$ instead of the velocity the first and last equations uncouple from the rest and may be ignored; and in terms of the Elsässer variables $\mathbf{P} = \mathbf{B} + \sqrt{\rho}\mathbf{V}$, $\mathbf{Q} = \mathbf{B} - \sqrt{\rho}\mathbf{V}$ the system becomes

$$\mathbf{P}_1 = (b+c)\mathbf{P},$$

$$\mathbf{Q}_1 = (b-c)\mathbf{Q},$$

$$(b+c)(b-c) = C. \tag{3}$$

The constant C may be analyzed by different arguments. Assume that U is submerged in Ω and that we wish the new equilibrium in U to connect smoothly with the old one in the rest of Ω. Since the total pressure is continuous in interfaces, necessarily $C = 1$. Another setting is as follows: assume, for this point only, that the density is constant (taken as 1) and that the new equilibrium is a product of the evolution of the old one after a slight perturbation. In mathematical terms, there exists a (heteroclinic) trajectory of the MHD dynamical system connecting both points. Provided $\mathbf{V}\cdot\mathbf{n}|_{\partial U} = \mathbf{B}\cdot\mathbf{n}|_{\partial U} = 0$, there are a number of integrals invariant by MHD evolution. These are the energy, the cross helicity, and the magnetic helicity. The last one is awkward to handle, but the conservation of the first two means

$$\int_U B_1^2 + V_1^2\, dV = \int_U B^2 + V^2 dV,$$

$$\int_U \mathbf{B}_1\cdot\mathbf{V}_1 dV = \int_U \mathbf{B}\cdot\mathbf{V} dV. \tag{4}$$

This means

$$\int_U (b+c)^2 P^2 dV = \int_U P^2 dV,$$

$$\int_U (b-c)^2 Q^2 dV = \int_U Q^2 dV. \tag{5}$$

Let us denote $\gamma = b + c$, and assume $C \neq 0$, so that γ never vanishes and $b - c = C/\gamma$. The first equation imposes a constraint on γ,

$$\int_U (\gamma^2 - 1) P^2 dV = 0, \tag{6}$$

while the second one determines, except for the sign, the value of C:

$$\int_U \left(\frac{C^2}{\gamma^2} - 1\right) Q^2 dV = 0. \tag{7}$$

The case $C = 0$ implies that the plasma is in an alfvénic state: $\mathbf{B}_1 = \pm\mathbf{V}_1$. This class of equilibria is invariant by the transformations in Eq. (1) and lies apart from others in an evolutionary sense, as described by the conservation of energy and cross helicity. We will exclude them from our consideration and assume that $C \neq 0$ is a constant, fixed by some argument, such as the ones above. Thus $\gamma(\psi) \neq 0$ for every level ψ.

Incidently, let us note that field-aligned equilibria are invariant for the transformations of Eq. (2): in particular any static equilibrium only yields field-aligned ones. For them Eq. (2) may be written as

$$\mathbf{B}_1 = k \cosh f(\psi)\mathbf{B},$$

$$\mathbf{V}_1 = k \sinh f(\psi)\mathbf{V}, \tag{8}$$

for some function f.

III. EXAMPLES

We will consider a few simple examples in order to show the usefulness of the transformations in Eq. (2) for generation of new equilibria. Let us consider the class of equilibria of the form

$$\mathbf{V} = (V_x(x,y), V_y(x,y), 0),$$

$$\mathbf{B} = (0, 0, B(x,y)), \tag{9}$$

where $\mathbf{V}$ and $\mathbf{B}$ satisfy

$$\nabla\cdot\mathbf{V} = 0,$$

$$\mathbf{V}\cdot\nabla\rho = 0,$$

$$\mathbf{V}\cdot\nabla B = 0,$$

$$\mathbf{V}\cdot\nabla\left(\frac{1}{2}V^2 + \frac{p}{\rho}\right) = 0. \tag{10}$$

It is known that conservation of the density and magnetic field size along streamlines and the Bernoulli law yield an ideal equilibrium. The magnetic surfaces are vertical cylinders whose sections with horizontal planes are the streamlines of the flow. Applying Eq. (2) with $a = 1$ yield

$$\mathbf{B}_1 = b\mathbf{B} + c\sqrt{\rho}\mathbf{V},$$

$$\mathbf{V}_1 = \frac{c}{\sqrt{\rho}}\mathbf{B} + b\mathbf{V}. \tag{11}$$

The new streamlines are not horizontal if $c \neq 0$, and the field lines are not vertical. The plasma flows at an angle that is constant in every magnetic surface, but may vary from one to the other. What is constant through the domain is the relation between the new and old electric fields: $\mathbf{V}_1 \times \mathbf{B}_1 = C\mathbf{V}\times\mathbf{B}$.

A general feature of the transformations of Eq. (2) is that one may obtain nonstatic equilibria from static ones, albeit always with field-aligned velocity. Thus, if we start with a classical toroidal equilibrium, say the Shafranov-Solov'ev one given by the flux function (in cylindrical coordinates (R,y,ϕ); see Ref. [2])

$$\psi(R,y)=R^2-y^2-(1-R^2)^2,$$

$$\mathbf{B}(R,\psi)=\frac{1}{R}\frac{\partial\psi}{\partial y}e_R-\frac{1}{R}\frac{\partial\psi}{\partial R}e_y+\frac{2\sqrt{\psi+1}}{R}e_\phi, \tag{12}$$

whose magnetic axis correspond to $\psi=5/4$ and the separatrix (where the surfaces cease to be toroids and intersect at the origin) is $\psi=-1$. Take as before $b=k\cosh f$, $c=k\sinh f$: we find a family of nonstatic equilibria

$$\mathbf{B}_1=k(\cosh f)\mathbf{B},$$

$$\mathbf{V}_1=k(\sinh f)\mathbf{B}. \tag{13}$$

f only needs to be constant along magnetic field lines, which allows for a large variety of flows. If we take in particular $f=f(\psi)$, this always holds. Take for simplicity $k=1$; since the hyperbolic cosine is an even function and reaches its minimum at zero, whereas the hyperbolic sine is an odd function, the new magnetic field will enhance the old one for any f, starting with it at the flux surface $\psi=0$ and making it larger symmetrically in ψ. The velocity is zero at $\psi=0$, while it takes opposite directions at both sides of this surface: the plasma flows along the field lines for positive (say) ψ and in the opposite directions for negative ψ. This change of direction of the flow at a certain magnetic surface is a feature uncommon in theoretical analysis.

Finally, let us mention that in Refs. [4,6] a different example appears and applications to astrophysical jets are claimed.

IV. ENERGY MINIMA

We will study which one of the equilibria in our class has minimal energy. That is, we need to minimize

$$E(\gamma)=\int_U B_1^2+\rho_1 V_1^2 dV$$

$$=\int_U P_1^2+Q_1^2 dV$$

$$=\int_U \gamma^2P^2+\frac{C^2}{\gamma^2}Q^2 dV. \tag{14}$$

C is fixed and γ is allowed to vary over the open set of continuous real functions γ defined in the interval of levels of U, $[\psi_1,\psi_2]$, which do not vanish anywhere in the interval.

It is intuitively obvious that E cannot have maxima: by taking γ large where P^2 is large, or small where Q^2 is large, we may obtain arbitrarily large energies. Let us find the critical points of the functional: denoting by (,) the scalar product in $L^2[\psi_1,\psi_2]$, we may write

$$E(\gamma)=(\gamma\mathbf{P},\gamma\mathbf{P})+\left(\frac{C}{\gamma}\mathbf{Q},\frac{C}{\gamma}\mathbf{Q}\right), \tag{15}$$

so that the differential in the direction of h is

$$E'(\gamma)(h)=(\gamma\mathbf{P},h\mathbf{P})+\left(\frac{C}{\gamma}\mathbf{Q},-\frac{Ch}{\gamma^2}\mathbf{Q}\right)$$

$$=\int_U h\left(\gamma P^2-\frac{C^2}{\gamma^3}Q^2\right)dV. \tag{16}$$

Let us use now a well-known theorem [7]: the level sets S_ψ are smooth surfaces for almost every ψ and for any continuous function G in U,

$$\int_U G|\nabla\psi|\,dV=\int_{\psi_1}^{\psi_2}d\psi\int_{S_\psi}G\,d\sigma, \tag{17}$$

where σ denotes the area measure. Assuming $\nabla\psi\neq\mathbf{0}$ (or simply $1/|\nabla\psi|$ integrable) we obtain

$$E'(\gamma)(h)=\int_{\psi_1}^{\psi_2}h(\psi)d\psi\int_{S_\psi}\frac{1}{|\nabla\psi|}\left(\gamma P^2-\frac{C^2}{\gamma^3}\right)d\sigma. \tag{18}$$

In the critical points of E, this integral must be zero for any h such that $\gamma+h\neq0$ anywhere. This obviously implies

$$\int_{S_\psi}\frac{1}{|\nabla\psi|}\left(\gamma P^2-\frac{C^2}{\gamma^3}\right)d\sigma=0, \tag{19}$$

for almost every ψ. Since, as assumed in the Introduction, this is a continuous function of ψ, the integral vanishes for all level sets. Hence

$$\gamma(\psi)^4\int_{S_\psi}\frac{P^2}{|\nabla\psi|}d\sigma=C^2\int_{S_\psi}\frac{Q^2}{|\nabla\psi|}d\sigma. \tag{20}$$

Since obviously γ and $-\gamma$ yield the same energy, this identity determines uniquely the possible energy minimum in our class of equilibria, provided none of these surface integrals vanish, which means that the plasma cannot be in an alfvénic state in the whole magnetic surface.

These solutions correspond really to minima of E: if we consider the second differential of the energy,

$$E'(\gamma)(h)=\left(\gamma P^2-\frac{C^2}{\gamma^3}Q^2,h\right),$$

$$E''(\gamma)(h,h)=\left(P^2+\frac{3C^2}{\gamma^4}Q^2,h^2\right)$$

$$=\int_U h^2\left(P^2+\frac{3C^2}{\gamma^4}Q^2\right)dV, \tag{21}$$

which is strictly positive for any nontrivial situation.

These minima may be identified as follows: since Eq. (14) means

$$\int_{S_\psi} \frac{P_1^2 - Q_1^2}{|\nabla \psi|} d\sigma = 0,$$

we have

$$\int_{S_\psi} \frac{\mathbf{B}_1 \cdot \mathbf{V}_1}{|\nabla \psi|} d\sigma = 0. \tag{22}$$

Thus they correspond to magnetic fields and velocities that are orthogonal in the mean in every magnetic surface.

General MHD equilibria are (linearly) stable if no small perturbation of them grows exponentially, although eventually they may drift away and their behavior becomes unknown. Linear stability is guaranteed if they are local minima of the energy (see, e.g., Ref. [8]). Were they strict minima, they would also stable in the long term; no solution close enough could leave the energy well. Such points, however, must be isolated in the phase space of magnetohydrodynamic evolution. Since any equilibrium state with a foliation of magnetic surfaces lies in a continuum of equilibria as determined by the transformations in Eq. (1), it cannot be a strict minimum, because there are other critical points arbitrarily close to it. If such strict minima exist at all, they certainly cannot be found by solving the Grad-Shafranov equation.

On the other hand, we have found strict minima within our class, which means that they are minima for the perturbations constant at every magnetic flux surface. One may ask if they are stable in the general sense, i.e., allowing for generic perturbations. Since every equilibrium state where $\mathbf{B} \cdot \mathbf{V} = 0$ everywhere satisfies the conditions of Eq. (16), the classical configurations of Sec. III with velocity and magnetic field orthogonal are minima in their class. Unfortunately not even the continuous spectrum needs to be stable for them [1]. We must conclude that perturbations within every magnetic surface may rend these equilibria unstable, as is indeed the case presented in Ref. [1].

Conversely, field-aligned equilibria $\mathbf{V} = \lambda(\psi)\mathbf{B}$ have a stable continuum [1], but they do not satisfy Eq. (16) unless they are static: $\lambda = 0$. If they are not minima in our restricted class, much less can they be so for general perturbations. Thus neither of these concepts implies the other.

V. RESISTIVE PLASMAS

Ideal MHD does not hold in the long run: there is always some resistivity and viscosity and therefore no real MHD equilibrium without some forcing. However, an ideal equilibrium decays only by the effects of ohmic dissipation and energy diffusion, which are small for most physically interesting plasmas, so that while the plasma evolution does not stop at ideal equilibria it should slow there. It seems worthwhile to find, at least among the restricted class of equilibria we are studying, which one has smaller dissipation and therefore lasts longer in a resistive setting. One could expect that minima of energy are good candidates for that. For this purpose we will assume that the density is constant (normalized to 1) and that the Prandtl number (quotient of resistivity and viscosity) is also 1. Calculations simplify in this important case (see Ref. [9]). We also assume that ∂U is formed by a single magnetic surface $\psi = \psi_1$, i.e., there is no hole within U where the foliation breaks down; this is indeed the usual case.

The dissipative term of the energy is, except by a positive factor,

$$\int_U \Delta \mathbf{V} \cdot \mathbf{V} + \Delta \mathbf{B} \cdot \mathbf{B}\, dV = \frac{1}{2} \int_{\partial U} \frac{\partial (V^2 + B^2)}{\partial n} d\sigma - \int_U |\nabla \mathbf{V}|^2 + |\nabla \mathbf{B}|^2 dV. \tag{23}$$

Thus the losses of energy are due to two factors: the flux of energy through ∂U and the ohmic heating within the domain. Since we wish to leave the field and velocity unchanged outside U, it seems reasonable to impose that the normal differentials of V_1^2 and V^2 must coincide at ∂U, and the same for the magnetic field. This means

$$\frac{\partial P_1^2}{\partial n} = \frac{\partial P^2}{\partial n},$$

$$\frac{\partial Q_1^2}{\partial n} = \frac{\partial Q^2}{\partial n}. \tag{24}$$

Since ∂U is formed by the magnetic surface $\psi = \psi_1$, and

$$\frac{\partial P_1^2}{\partial n} = 2\gamma(\psi_1)\gamma'(\psi_1) P^2 + \gamma(\psi_1)^2 \frac{\partial P^2}{\partial n} \tag{25}$$

(analogously with Q_1), the condition to impose upon γ is

$$\gamma(\psi_1) = 1,$$

$$\gamma'(\psi_1) = 0. \tag{26}$$

The alternative $\gamma(\psi_1) = -1$ reduces to this by changing $\gamma \to -\gamma$. Now we must minimize the functional

$$D(\gamma) = \int_U |\nabla \mathbf{P}_1|^2 + |\nabla \mathbf{Q}_1|^2 dV = \int_U |\nabla(\gamma \mathbf{P})|^2 + \left| \nabla \left(\frac{C}{\gamma} \mathbf{Q} \right) \right|^2 dV, \tag{27}$$

when γ belongs to the space of functions satisfying the above boundary conditions. Perturbations h such that $\gamma + h$ lies also in this space must satisfy that both the values of h and h' at ψ_1 are zero. The differential of D at γ in the direction of h is

$$D'(\gamma)(h) = (\nabla(\gamma \mathbf{P}), \nabla(h\mathbf{P})) + C^2 \left(\nabla \left(\frac{1}{\gamma} \mathbf{Q} \right), \nabla \left(-\frac{h}{\gamma^2} \mathbf{Q} \right) \right). \tag{28}$$

By assuming that h satisfies the boundary conditions above and using Gauss's theorem,

$$D'(\gamma)(h)=\int_U h\left[\mathbf{P}\cdot\Delta(\gamma\mathbf{P})-\frac{C^2}{\gamma^2}\mathbf{Q}\cdot\Delta\left(\frac{1}{\gamma}\mathbf{Q}\right)\right]dV$$
$$=\int_{\psi_0}^{\psi_1}h(\psi)d\psi\int_{S_\psi}\frac{1}{|\nabla\psi|}\left[\mathbf{P}\cdot\Delta(\gamma\mathbf{P})-\frac{C^2}{\gamma^2}\mathbf{Q}\cdot\Delta\left(\frac{1}{\gamma}\mathbf{Q}\right)\right]d\sigma. \quad (29)$$

ψ_0 corresponds to some magnetic axis or point. Since the range of possible h is large enough, the surface integral must vanish for every ψ. This is a nonlinear second-order differential equation in γ, obtained by developing the Laplacians. The initial conditions at ψ_1 are as stated $\gamma(\psi_1)=1$, $\gamma'(\psi_1)=0$, so that solutions exist at least in some interval $(\psi_-,\psi_1]$. Since the equations are nonlinear, there is, in principle, no guarantee that they extend up to ψ_0, but it is not difficult to see that this is indeed the case. The equation satisfied by γ may be abbreviated to

$$L_1(\gamma)=L_2(1/\gamma), \quad (30)$$

where L_1 and L_2 are linear differential operators. It is known that if the solution cannot be extended past ψ_-, because it blows up there: $|\gamma(\psi_-)|\to\infty$. This means that $1/\gamma$ and therefore $L_2(1/\gamma)$ remain bounded at ψ_-. Writing $L_2(1/\gamma)=F$, the equation becomes $L_1(\gamma)=F$, for a bounded independent term F; since all the terms of the equation are well behaved near ψ_-, and L_1 is linear, the solution cannot blow up at ψ_-, against the first hypothesis. Thus γ is defined in the whole interval $(\psi_0,\psi_1]$. That γ cannot vanish anywhere follows from a similar argument, because then $1/\gamma$ would blow up while $L_1(\gamma)$ would remain smooth, against the properties of linear equations. That the solution behaves well at the axis ψ_0 follows from a study of the form of the equation coefficients, which are integrals in the magnetic surfaces, that we will not detail here. Thus there exist really unique solutions yielding critical points of the functional D. It is not as simple as in the energy case to prove that they correspond to minima, but this is strongly supported by the form of the positive functional D; certainly rapidly varying γ yield arbitrarily large dissipation, so that there is no chance of D having a maximum. Anyway, if $\mathbf{P}_1$, $\mathbf{Q}_1$ is a critical point of D, necessarily

$$\int_{S_\psi}\frac{\mathbf{P}_1\cdot\Delta\mathbf{P}_1-\mathbf{Q}_1\cdot\Delta\mathbf{Q}_1}{|\nabla\psi|}d\sigma=0, \quad (31)$$

which means

$$\int_{S_\psi}\frac{\mathbf{B}_1\cdot\Delta\mathbf{V}_1+\mathbf{V}_1\cdot\Delta\mathbf{B}_1}{|\nabla\psi|}d\sigma=0. \quad (32)$$

Static equilibria satisfy Eq. (27), as well as those, detailed in Sec. III, of the form $\mathbf{V}=(V_1(x,y),V_2(x,y),0)$, $\mathbf{B}=(0,0,B_3(x,y))$. These, as stated, can be unstable even in the continuous spectrum. Field-aligned velocities do not in general satisfy Eq. (27), in spite of the stability of their continuum. We see again that none of these concepts is equivalent, nor implies any other.

VI. CONCLUSIONS

The study of the stability of ideal magnetohydrodynamic equilibria has a long and distinguished history. The most basic notion, linear stability, considers the solutions of the MHD system linearized around a state of equilibrium, where all the magnitudes are time independent. The system is unstable if there are exponentially growing solutions, marginally stable otherwise; the most stringent condition of stability demands all solutions to decay, meaning that small perturbations of the equilibrium will tend to disappear. The addition of a small resistivity to the ideal MHD equations introduces the new class of the so-called resistive instabilities, which are certainly more realistic than ideal ones.

It was early recognized that an equilibrium which is a strict minimum of the total (kinetic plus magnetic) energy would be stable, and this provided some sufficient criteria for stability. Unfortunately, equilibria simple enough to be expressed analytically are few, even in the restricted class of axisymmetric static ones, when they satisfy the more manageable Grad-Shafranov equation. Thus, more often than not, all criteria of stability must be studied numerically.

Recently, a simple study of the symmetries of the MHD system has yielded a method for generating whole families of ideal equilibria starting from one provided with a foliation of magnetic surfaces, i.e., surfaces everywhere tangential to the velocity and magnetic field vectors: most classical equilibria satisfy this condition, which is also very desirable for magnetic confinement. These families keep the same foliation of the original equilibrium, but both velocity and field may vary widely otherwise. This paper begins by providing a simpler formulation, using Elsässer variables, of the tranformation mentioned above, which will be useful later, and showing with a number of examples the rather surprising properties of equilibria one may obtain from classical, usually static, ones. However, our main purpose is, by using the family of equilibria created using this process, the comparison of the different notions of stability.

We work with three concepts: linearly stable equilibria, those which are energy minima among our restricted family, and those equilibria which minimize ohmic and viscous losses when some small diffusivity is added, presumably lasting longer under these slightly resistive conditions. Our intuition tends to think that these properties are roughly equivalent. The variational calculus needed to calculate these extrema is far easier than the general one because of the more restricted class of equilibria under study. The results are somewhat surprising: none of these notions implies any other, so that, for example, a minimum (in our class) of energy may be linearly unstable, thus highlighting how demanding is the condition of maintaining the magnetic surfaces intact. Neither is this minimum more efficient at minimizing ohmic losses than other members of the family. Also, equilibria with a stable continuous spectrum may not be energy minima even within our family. The examples showing that these concepts are different are very simple: static equilibria, equilibria where magnetic field and velocity are orthogonal, and those where they are aligned are enough to provide counterexamples. It is hoped that these results may be conceptually useful to clarify the different notions of stability; more generally, this family of equilibria could prove useful to test other magnetohydrodynamic properties.

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